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LETTER TO THE EDITOR

Series studies of the four-state Potts model†

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Abstract. The four-state Potts model is studied via series expansions. Free energy and susceptibility high-temperature series are calculated for hypercubic lattices of general dimensionality. Estimates of critical temperatures and critical indices γ and α are obtained as functions of dimensionality. The free energy, magnetisation, and susceptibility low-temperature series are calculated for $d = 4$. The high- and low-temperature series together indicate that the phase transition in this dimensionality is first order. From this result we would suggest that the apparent second-order transitions shown by the susceptibility for $d \geqslant 4$ might be explained as spinodal pseudotransitions.

The four-state Potts model has received much attention recently. It is a possible model for the behaviour of some classes of adsorbed gases on graphite, for example see Domany *et al* (1978). As one of the q-state Potts models and as a special case of the Ashkin–Teller model, much is known about it, particularly in two dimensions, for example from work by Wu and Lin (1974), Kim and Joseph (1975), and Enting (1975).

An interesting question is the nature of the transition as a function of d and q. In $d = 2$ Baxter (1973) showed that $q = 4$ is the highest q for which the phase transition is second order. Evidence from ϵ expansions around $d = 6$ by Priest and Lubensky (1976) would support a first-order transition starting possibly at $d = \frac{10}{3}$ for $q \geqslant \frac{10}{3}$.

We are in the process of investigating the general q-state Potts models via high- and low-temperature series expansions. We developed high-temperature series for the Ashkin–Teller model free energy and susceptibility on hypercubic lattices. In this letter we present only the series for the special case of the Potts model and the results obtained upon analysing these series. Results for the general Ashkin–Teller model will be reported elsewhere. The Hamiltonian for the Ashkin–Teller model can be written as

$$-\mathscr{H}_{\text{A-T}} = J_2 \sum_{\langle ij \rangle} (S_iS_j + \sigma_i\sigma_j) + J_4 \sum_{\langle ij \rangle} S_iS_j\sigma_i\sigma_j \tag{1}$$

where on each lattice site i, the two Ising variables S_i and σ_i can independently take the values ± 1, and $\langle ij \rangle$ as usual denotes nearest neighbours. The four-state Potts model is obtained when $J_2 = J_4$. The Hamiltonian is then given by

$$\mathscr{H}_{\text{Potts}} = -J \sum_{\langle ij \rangle} (S_iS_j - \tfrac{1}{4}) \tag{2}$$

where $J = 4J_2$.

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Following earlier work of Ditzian (1972a, b) we linearise the partition function using the variables

$$Y=(v+wv)/(1+wv^2) \qquad \text{and} \qquad Z=(w+v^2)/(1+wv^2)$$

where $v=\tanh \beta J_2$ and $w=\tanh \beta J_4$, to obtain an expression for the susceptibility:

$$\tfrac{1}{2}\chi=\tfrac{1}{2}+T'_r\sum_{l>k}\prod_{\langle ij\rangle}[1+Y(S_iS_j+\sigma_i\sigma_j)+Z\sigma_i\sigma_jS_iS_j]\sigma_k\sigma_l \tag{3}$$

and a similar expression for the free energy. The prime on the trace indicates that we take only the terms linear in N.

We expand (3) graphically using the primitive method described in Domb and Green (1974). Each bond can be covered by either one of the Y terms or the Z term. The skeleton graphs occurring in the expansion include disconnected graphs in addition to those used by Fisch and Harris (1977) and Ditzian and Kadanoff (1979).

It is necessary to construct all the coverings of each skeleton graph. The weight of each covering is simply Y^nZ^m. For the Potts series we set $Y=Z$.

Some series are presented in tables 1 and 2, others are available on request. The free energy agrees with the latest two-dimensional q-state Potts series of Enting (1977). The susceptibility agrees with the shorter series in Ditzian (1972a, b) apart from one term in the three-dimensional calculation where $4560\,Y^5Z^2$ should have been $5520\,Y^5Z^2$. The susceptibility agrees with the existing terms in the series of Kim and Joseph (1975) on the square lattice.

The series for the susceptibility gave reasonably well converged estimates of v_c and γ (table 3). We might expect reasonably good results even for the higher dimension since the susceptibility graphs span up to ten dimensions. In contrast, while the specific heat

Table 1. The free energy coefficients b_n where $\ln 4+4\ln(\cosh^3 K+\sinh^3 K)+\Sigma_{n=4}\, b_nV^n$.

d \ n	4	5	6	7	8	9	10	11
3	9	36	120	504	2200·5	9804	45 954	221 112
4	18	72	336	1728	9981	57 624	359 412	2271 552

Table 2. The susceptibility series coefficients a_n where $\chi=\frac{1}{2}+\Sigma_{n=1}\, a_nV^n$.

n	$d=3$	$d=4$
1	3	4
2	18	32
3	105	252
4	636	2032
5	3807	16 292
6	23 094	132 000
7	140 469	1070 716
8	857 736	8729 216
9	5251 163	71 320 324
10	32 230 218	584 550 656

Table 3. Estimates of critical temperatures and indices from the susceptibility and specific heat high-temperature series for dimensionality d.

	$d=2$	2·5	3	$\frac{10}{3}$	4	5	6	7	8	10
V_c^γ	0·267	0·197	0·160	0·1435	0·119	0·0968	0·0815	0·0704	0·06188	0·0497
	±0·003	±0·005	±0·003	±0·0020	±0·002	±0·001	±0·0010	±0·0004	±0·00001	±0·0001
γ	1·17	0·92	0·84	0·82	0·76	0·85	0·90	0·943	0·965	0·923
	±0·08	±0·05	±0·01	±0·03	±0·02	±0·02	±0·02	±0·01	±0·002	±0·005
$V_c^{(\alpha)}$	0·260	0·168	0·162	0·148	0·120	0·102	0·090			
	±0·030	±0·015	±0·010	±0·010	±0·007	±0·005	±0·005			
α	0·50	0·76	0·68	0·67	0·61	0·66	0·66			
	±0·05	±0·10	±0·10	±0·10	±0·07	±0·05	±0·05			

series goes up to eleventh order, its graphs can be only five dimensional. For this reason, the higher dimensional ($d>5$) specific heats are suspect. In fact, the estimates from the specific heat series were rather badly converged in all dimensions.

The critical temperature v_c shows a smooth d-dependence with no untoward behaviour at $d=\frac{10}{3}$ or anywhere else. But there seems to be a minimum in γ somewhere near four dimensions which could possibly be interpreted as an indication of a 'tricritical' dimension or other anomalous behaviour. Note, however, that there is a discrepancy between estimates of v_c from the susceptibility, v_c^γ, and that from the specific heat, v_c^α. (v_c^α is much below v_c^γ at $d=2, 2\cdot5$ and somewhat above it at $d=5, 6$.) The estimates of α are taken at v_c^α as at v_c^γ they were not well converged. In five and six dimensions the susceptibility series has a very consistent pair of poles at $v=\pm0\cdot14$ which we do not understand.

Confluent singularities (Rudnick and Nelson 1976) could well shift the estimates by a considerable amount and $\gamma=1$ is therefore a possibility for a large range of dimensions. Since the 4-state Potts model in $d=2$ is a special point where the Ashkin–Teller line of singularities splits into two (Wu 1974), it is quite likely that crossover corrections affect the series at that dimensionality.

The obvious next step is to approach the transition from below as the high-temperature series could have crossed the first-order transition into the metastable region and be showing us the transition from the metastable to the unstable state at the spinodal line. This spinodal line might have an approximate meaning in our series expansion even though (Langer 1973) its real existence in an exact theory is problematical.

The low-temperature series must be attacked separately for each dimension. We picked $d=4$ as a dimension where one might expect the transition to be first order and where the graphical data is available from Sykes (1979) Ising calculations. The preliminary results for the low-temperature series are given in table 4. The series are

Table 4. The low-temperature polynomials L_n in the standard notation up to 35th power of z.

$$L_1=3z^8$$
$$L_2=12z^{14}+24z^{15}-40\cdot5z^{16}$$
$$L_3=84z^{20}+336z^{21}-240z^{22}-1152z^{23}+981z^{24}$$
$$L_4=18z^{24}+900z^{26}+4248z^{27}+810z^{28}-24\,552z^{29}-3780z^{30}+52\,488z^{31}-30\,152\cdot25z^{32}$$
$$L_5=432z^{30}+864z^{31}+9726z^{32}+62\,256z^{33}+42\,480z^{34}-378\,240z^{35}+\dots$$
$$L_6=180z^{34}+0z^{35}+\dots$$

obtained by a method similar to that described by Miyashita *et al* (1979). However, since we needed only a few lattice constants we calculated them directly rather than work with the codes.

The low-temperature series are in the variable $z = \exp(-4J/kT)$. Padé approximants of $\partial \ln M/\partial z$ give estimates of the low-temperature critical point which is $z_c^L = 0{\cdot}647 \pm 0{\cdot}010$. Note that this is slightly above the estimate $z_c^H = 0{\cdot}621 \pm 0{\cdot}005$ from the high-temperature susceptibility. If z_c^L were indeed significantly the higher of the two then we would interpret the result in terms of spinodal points and a first-order phase transition. This interpretation is strengthened by the analysis of M itself. Estimates of M indicate (as shown in figure 1) that M is definitely non-zero in the range of z where the transition occurs.

Additional evidence for the first-order transition can be seen in figure 2 where the Padé estimates for the free energy curves are shown. They seem to cross with different slopes at $z_c = 0{\cdot}635 \pm 0{\cdot}005$.

We think that the evidence is that at $d = 4$ the $q = 4$ Potts model undergoes a first-order phase transition at $z_c = 0{\cdot}635 \pm 0{\cdot}005$ and that there are spinodal pseudo-

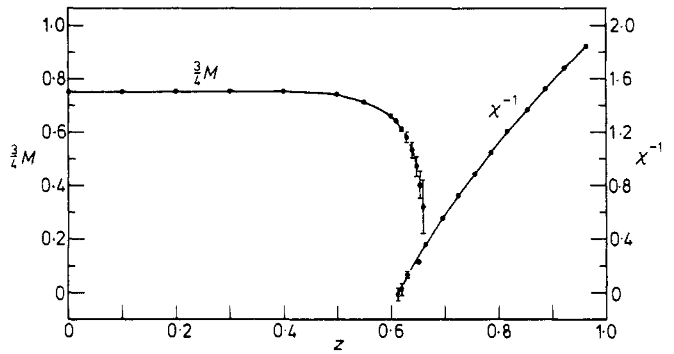

Figure 1. The inverse of the high-temperature susceptibility and the low-temperature magnetisation versus z, the low-temperature variable. Error bars are shown when the apparent errors are large enough to plot.

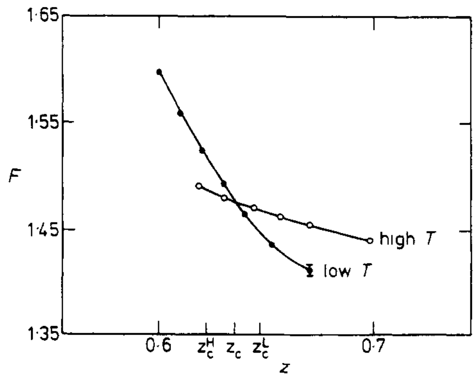

Figure 2. The low-temperature and high-temperature free energies versus z in the neighbourhood of the transition. Only one error bar is shown in this figure since the apparent errors are smaller than the data points with that one exception.

transitions at $z_c^H = 0{\cdot}621 \pm 0{\cdot}005$ and $z_c^L = 0{\cdot}647 \pm 0{\cdot}010$ with apparent $\gamma = 0{\cdot}76 \pm 0{\cdot}02$ and $\gamma' = 0{\cdot}72 \pm 0{\cdot}10$.

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